Global fitting

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Outline

- Motivation
- Mathematical foundation (Yang) Yang’s note
- Example1 (Xiu): The results used in BESIII Ying-Hui Guan *et al* 2013 *Chinese Phys. C* **37** 106201
- Example2 (Min): A Classical example
- Code implementation (Yan)

Motivation

- ✓ Sometimes we use the DT method or ST method and have many tagged modes, then we need to consider the contaminate effect from other tagged modes and the contribution from peaking background, then we will consider whether there is a general method to solve this problem and give us a good fitting results?
- ✓ Yes, this global fitting method is used in many experiments.

Decay modes	separately calculated $\mathcal{B}$	global fit $\mathcal{B}$	PDG $\mathcal{B}$ [1]	Belle $\mathcal{B}$ [6]
pK_S	$1.58 \pm 0.16 \pm 0.03$	$1.52 \pm 0.08 \pm 0.03$	1.15 ± 0.30	
$pK^-\pi^+$	$6.15 \pm 0.33 \pm 0.25$	$5.84 \pm 0.27 \pm 0.23$	5.0 ± 1.3	$6.84 \pm 0.24^{+0.21}_{-0.27}$
$pK_S\pi^0$	$1.89 \pm 0.32 \pm 0.05$	$1.87 \pm 0.13 \pm 0.05$	1.65 ± 0.50	
$pK_S\pi^+\pi^-$	$1.44 \pm 0.25 \pm 0.09$	$1.53 \pm 0.11 \pm 0.09$	1.30 ± 0.35	
$pK^-\pi^+\pi^0$	$4.73 \pm 0.38 \pm 0.33$	$4.53 \pm 0.23 \pm 0.30$	3.4 ± 1.0	
$\Lambda\pi^+$	$1.38 \pm 0.18 \pm 0.04$	$1.24 \pm 0.07 \pm 0.03$	1.07 ± 0.28	
$\Lambda\pi^+\pi^0$	$5.93 \pm 0.77 \pm 0.19$	$7.01 \pm 0.37 \pm 0.19$	3.6 ± 1.3	
$\Lambda\pi^+\pi^-\pi^+$	$4.42 \pm 0.59 \pm 0.23$	$3.81 \pm 0.24 \pm 0.18$	2.6 ± 0.7	
$\Sigma^0\pi^+$	$1.12 \pm 0.18 \pm 0.04$	$1.27 \pm 0.08 \pm 0.03$	1.05 ± 0.28	
$\Sigma^+\pi^0$	$1.13 \pm 0.23 \pm 0.03$	$1.18 \pm 0.10 \pm 0.03$	1.00 ± 0.34	
$\Sigma^+\pi^+\pi^-$	$3.70 \pm 0.42 \pm 0.18$	$4.25 \pm 0.24 \pm 0.20$	3.6 ± 1.0	
$\Sigma^+\omega$	$1.43 \pm 0.33 \pm 0.12$	$1.56 \pm 0.20 \pm 0.07$	2.7 ± 1.0	
$N_{\Lambda_c^+\bar{\Lambda}_c^-} = (105.9 \pm 4.8 \pm 0.5) \times 10^3$				

Absolute measurements of hadronic branching fractions of Λ_c^+ baryon

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Mathematical part (Yang)

- ✓ Expected value
- ➢ Some basic equation

$$\begin{cases} \eta & a\ vector\ of\ N\ observables \\ y & N\ measurements\ of\ \eta\ with\ errors\ contained \\ \zeta & J\ unmeasurable\ variables\ (\zeta_1, \cdots, \zeta_J) \end{cases} \quad (1)$$

We also have K constraint equations during η and ζ.

$$f_k(\eta_1, \cdots, \eta_N, \zeta_1, \cdots, \zeta_J) = 0 \quad (2)$$

$$\chi^2 = (y - \eta)^T V^{-1}(y)(y - \eta) = minimum \quad (3)$$

We use Lagrangian parameter method to search the solutions.

$$\chi^2(\eta, \zeta, \lambda) = (y - \eta)^T V^{-1}(y)(y - \eta) + 2\lambda^T f(\eta, \zeta) = minimum \quad (5)$$

We can easily find if we have a group of solutions of (3) and (4), this group must also be the solutions of (5). For simplicity, we assume that (5) only contains one group of solutions then the solutions (3) and (4) must be the same as (5). If (5) have many groups of solutions, we can find all solutions and take them into (3) and (4) to find the real solutions.

$$\nabla_\eta \chi^2 = -2V^{-1}(y)(y - \eta) + 2(\frac{\partial f_k(\eta, \zeta)}{\partial \eta_i})^T \lambda_k = 0 \quad (6)$$

$$\nabla_\zeta \chi^2 = 2(\frac{\partial f_k(\eta, \zeta)}{\partial \zeta_i})^T \lambda_k = 0 \quad (7)$$

$$\nabla_\lambda \chi^2 = 2f(\eta, \zeta) = 0 \quad (8)$$

$$(F_\eta)_{ki} = \frac{\partial f_k}{\partial \eta_i}, \ (F_\zeta)_{kj} = \frac{\partial f_k}{\partial \zeta_j} \quad (9)$$

$$V^{-1}(y)(\eta - y) + F_\eta^T \lambda = 0 \tag{10}$$

$$F_\zeta^T \lambda = 0 \tag{11}$$

$$f(\eta, \zeta) = 0 \tag{12}$$

- Now make iterative equation.

We expand Eq (12) at point (η^v, ζ^v) using the Taylor formula like this.

$$f_k(\eta,\zeta) = f_k^v + (\frac{\partial f_k}{\partial \eta_i})^v(\eta_i - \eta_i^v) + (\frac{\partial f_k}{\partial \zeta_j})^v(\zeta_j - \zeta_j^v) + \cdots \tag{13}$$

$$f_k^v + (\frac{\partial f_k}{\partial \eta_i})^v(\eta_i - \eta_i^v) + (\frac{\partial f_k}{\partial \zeta_j})^v(\zeta_j - \zeta_j^v) + \cdots = 0 \tag{14}$$

$$f_k^v + (\frac{\partial f_k}{\partial \eta_i})^v(\eta_i - \eta_i^v) + (\frac{\partial f_k}{\partial \zeta_j})^v(\zeta_j - \zeta_j^v) = 0 \tag{15}$$

- $$f_k^v + F_\eta^v(\eta^{v+1} - \eta^v) + F_\zeta^v(\zeta^{v+1} - \zeta^v) = 0 \tag{16}$$
- $$V^{-1}(y)(\eta^{v+1} - y) + (F_\eta^T)^v \lambda^{v+1} = 0 \tag{17}$$
- $$(F_\zeta^T)^v \lambda^{v+1} = 0 \tag{18}$$

- Do some calculate to separate two parts

We can simplify Eq(17) like this.

$$\eta^{v+1} = y - V(y)(F_\eta^T)^v \lambda^{v+1} \tag{19}$$

Take Eq(19) into Eq(16), we have.

$$f^v + F_\eta^v(y - V(y)(F_\eta^T)^v\lambda^{v+1} - \eta^v) + F_\zeta^v(\zeta^{v+1} - \zeta^v) = 0 \tag{20}$$

$$f^v + F_\eta^v(y - \eta^v) - F_\eta^v V(y)(F_\eta^T)^v \lambda^{v+1} + F_\zeta^v(\zeta^{v+1} - \zeta^v) = 0 \tag{21}$$

$$r = f^v + F_\eta^v(y - \eta^v) \tag{22}$$

$$s = F_\eta^v V(y)(F_\eta^T)^v \tag{23}$$

$$V^{-1}(y)(\eta^{v+1} - y) + (F_\eta^T)^v \lambda^{v+1} = 0 \quad (17)$$

$$(F_\zeta^T)^v \lambda^{v+1} = 0 \quad (18)$$

$$r + F_\zeta^v(\zeta^{v+1} - \zeta^v) = s\lambda^{v+1} \quad (24)$$

$$(F_\zeta^T)^v s^{-1}[r + F_\zeta^v(\zeta^{v+1} - \zeta^v)] = 0 \quad (25)$$

$$(F_\zeta^T)^v s^{-1} r + (F_\zeta^T)^v s^{-1} F_\zeta^v \zeta^{v+1} - (F_\zeta^T)^v s^{-1} F_\zeta^v \zeta^v = 0 \quad (26)$$

$$r = f^v + F_\eta^v(y - \eta^v) \quad (22)$$

$$s = F_\eta^v V(y)(F_\eta^T)^v \quad (23)$$

- Iterative formulas

- $$\zeta^{v+1} = \zeta^v - [(F_\zeta^T)^v s^{-1} F_\zeta^v]^{-1}(F_\zeta^T)^v s^{-1} r \quad (27)$$
- $$\lambda^{v+1} = s^{-1}[r + F_\zeta^v(\zeta^{v+1} - \zeta^v)] \quad (28)$$
- $$\eta^{v+1} = y - V(y)(F_\eta^T)^v \lambda^{v+1} \quad (29)$$

✓ Statistical errors

- Some basic equation

$$\hat{\eta} = g(y) \quad (30)$$

$$\hat{\zeta} = h(y) \quad (31)$$

Take Eq(27) and Eq(28) into Eq(29), we have

$$\eta^{v+1} = y - V(y)(F_\eta^T)^v s^{-1}[I_N - (F_\zeta)^v[(F_\zeta^T)^v s^{-1} F_\zeta^v]^{-1}(F_\zeta^T)^v s^{-1}] \cdot (f^v + F_\eta^v(y - \eta^v)) = g(y) \quad (32)$$

From Eq(27), we have,

$$\zeta^{v+1} = \zeta^v - [(F_\zeta^T)^v s^{-1} F_\zeta^v]^{-1}(F_\zeta^T)^v s^{-1}(f^v + F_\eta^v(y - \eta^v)) = h(y) \quad (33)$$

- Some supplemental calculations

We have know that,

$$V_{kl}(y) = E(\ (y_k(x) - E(y_k(x)))(y_l(x) - E(y_l(x)))\ ) \quad (36)$$

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$$V_{kl}(y) = E(\ (y_k(x) - E(y_k(x)))(y_l(x) - E(y_l(x)))\ ) \tag{36}$$

$$y_k(x) = y_k(\mu) + \sum_i \frac{\partial y_k}{\partial x_i}|_{x_i=\mu_i}(x_i - \mu_i) + \cdots \tag{37}$$

$$E(y_k(x)) = y_k(\mu) \tag{38}$$

$$V_{kl}(y) = E((\sum_i \frac{\partial y_k}{\partial x_i}|_{x=\mu}(x_i - \mu_i))(\sum_j \frac{\partial y_l}{\partial x_j}|_{x=\mu}(x_j - \mu_j))) \tag{39}$$

$$= \sum_i \sum_j \frac{\partial y_k}{\partial x_i}|_{x=\mu} \frac{\partial y_l}{\partial x_j}|_{x=\mu} E((x_i - \mu_i)(x_j - \mu_j))$$

$$= \sum_i \sum_j \frac{\partial y_k}{\partial x_i}|_{x=\mu} V_{ij} \frac{\partial y_l}{\partial x_j}|_{x=\mu} \tag{40}$$

$$S_{ki} = \frac{\partial y_k}{\partial x_i}|_{x=\mu} \tag{41}$$

Finally, we have,

$$V_{kl}(y) = S_{ki} V_{ij} (S_{lj})^T \tag{42}$$

$$V(y) = SV(x)S^T \tag{43}$$

- Calculate statistical errors

$$Cov(\hat{\eta}, \hat{\zeta})_{kl} = E((\hat{\eta}_k - E(\hat{\eta}_k))(\hat{\zeta}_l - E(\hat{\zeta}_l))) \tag{44}$$

$$= E(\sum_i \frac{dg_k}{dy_i}|_{y=\eta_0}(y_i - \eta_{i0}) \sum_j \frac{dh_l}{dy_j}|_{y=\eta_0}(y_j - \eta_{j0})) \tag{45}$$

$$= \sum_i \sum_j \frac{dg_k}{dy_i}|_{y=\eta_0} \frac{dh_l}{dy_j}|_{y=\eta_0} E((y_i - \eta_{i0})(y_j - \eta_{j0})) \tag{46}$$

$$= \frac{dg_k}{dy_i}|_{y=\eta_0} V_{ij}(y) \frac{dh_l}{dy_j}|_{y=\eta_0} \tag{47}$$

$$\hat{\eta} = g(y) \tag{30}$$

$$\hat{\zeta} = h(y) \tag{31}$$

We expand $\hat{\eta}_i$ at the expected point of η_{j0},

$$\hat{\eta}_i = g_i(\eta_0) + \sum_j (y_j - \eta_{j0}) \frac{\partial g_i}{\partial y_j}|_{y_j=\eta_{j0}} + \cdots \tag{34}$$

We get this result,

$$E(\hat{\eta}_i) = g_i(\eta_0) \tag{35}$$

- $$Cov(\hat{\eta}, \hat{\zeta}) = (\frac{dg}{dy})V(y)(\frac{dh}{dy})^T \quad (48)$$

Similarly,

- $$V(\hat{\eta}) = (\frac{dg}{dy})V(y)(\frac{dg}{dy})^T \quad (49)$$
- $$V(\hat{\zeta}) = (\frac{dh}{dy})V(y)(\frac{dh}{dy})^T \quad (50)$$
- $$Cov(\hat{\eta}, \hat{\zeta}) = (\frac{dg}{dy})V(y)(\frac{dh}{dy})^T \quad (51)$$

- Do more calculations to find some properties.

$$g(y) = y - V(y)(F_\eta^T)^v s^{-1}[I_N - (F_\zeta)^v[(F_\zeta^T)^v s^{-1} F_\zeta^v]^{-1}(F_\zeta^T)^v s^{-1}]$$
$$\cdot(f^v + F_\eta^v(y - \eta^v)) \quad (52)$$
$$h(y) = \zeta^v - [(F_\zeta^T)^v s^{-1} F_\zeta^v]^{-1}(F_\zeta^T)^v s^{-1}(f^v + F_\eta^v(y - \eta^v)) \quad (53)$$

$$\frac{dg}{dy} = I - VF_\eta^T s^{-1}[I_N - F_\zeta[F_\zeta^T s^{-1} F_\zeta]^{-1} F_\zeta^T s^{-1}]F_\eta \quad (54)$$
$$= I - V[[F_\eta^T s^{-1} F_\eta] - [F_\eta^T s^{-1} F_\zeta][F_\zeta^T s^{-1} F_\zeta]^{-1}[F_\zeta^T s^{-1} F_\eta]] \quad (55)$$
$$\frac{dh}{dy} = -[F_\zeta^T s^{-1} F_\zeta]^{-1}[F_\zeta^T s^{-1} F_\eta] \quad (56)$$
$$G = F_\eta^T s^{-1} F_\eta, \; H = F_\eta^T s^{-1} F_\zeta, \; U^{-1} = F_\zeta^T s^{-1} F_\zeta \quad (57)$$

$$s = F_\eta V(y) F_\eta^T \quad (58)$$

$$G = F_\eta^T s^{-1} F_\eta = V^{-1}(y) \quad (59)$$
$$H = F_\eta^T s^{-1} F_\zeta = V^{-1}(y) F_\eta^{-1} F_\zeta \quad (60)$$
$$s^T = F_\eta V^T(y) F_\eta^T = F_\eta V(y) F_\eta^T = s \quad (62)$$
$$s^{-1} = (s^{-1})^T \quad (63)$$

- $$s^{-1} = (s^{-1})^T \tag{63}$$
- $$G = F_\eta^T s^{-1} F_\eta = V^{-1}(y) \tag{64}$$
- $$H = F_\eta^T s^{-1} F_\zeta = V^{-1}(y) F_\eta^{-1} F_\zeta \tag{65}$$
- $$U^{-1} = F_\zeta^T s^{-1} F_\zeta \tag{66}$$
- $$H^T = F_\zeta^T s^{-1} F_\eta \tag{67}$$
- $$G^T = G \tag{68}$$
- $$(U^{-1})^T = U^{-1} \tag{69}$$
- $$U = (F_\zeta)^{-1} s (F_\zeta^T)^{-1} \tag{70}$$

$$HUH^T = V^{-1}(y) F_\eta^{-1} F_\zeta (F_\zeta)^{-1} s (F_\zeta^T)^{-1} F_\zeta^T (F_\eta^T)^{-1} V^{-1}(y) \tag{73}$$

$$= V^{-1}(y) F_\eta^{-1} F_\eta V(y) F_\eta^T (F_\eta^T)^{-1} V^{-1}(y)$$

$$= V^{-1}(y) \tag{74}$$

$$\downarrow$$

$$G = HUH^T = V^{-1}(y) \tag{75}$$

- $$\frac{dg}{dy} = I_N \tag{76}$$
- $$\frac{dh}{dy} = -UH^T \tag{77}$$

$$\frac{dg}{dy} = I - V F_\eta^T s^{-1} [I_N - F_\zeta [F_\zeta^T s^{-1} F_\zeta]^{-1} F_\zeta^T s^{-1}] F_\eta \tag{54}$$

$$= I - V[[F_\eta^T s^{-1} F_\eta] - [F_\eta^T s^{-1} F_\zeta][F_\zeta^T s^{-1} F_\zeta]^{-1}[F_\zeta^T s^{-1} F_\eta]] \tag{55}$$

$$\frac{dh}{dy} = -[F_\zeta^T s^{-1} F_\zeta]^{-1}[F_\zeta^T s^{-1} F_\eta] \tag{56}$$

$$G = F_\eta^T s^{-1} F_\eta, \ H = F_\eta^T s^{-1} F_\zeta, \ U^{-1} = F_\zeta^T s^{-1} F_\zeta \tag{57}$$

$$\downarrow$$

$$\frac{dg}{dy} = I_N - V[G - HUH^T] \tag{71}$$

$$\frac{dh}{dy} = -UH^T \tag{72}$$

$$V(\hat{\eta}) = (\frac{dg}{dy}) V(y) (\frac{dg}{dy})^T \tag{49}$$

$$V(\hat{\zeta}) = (\frac{dh}{dy}) V(y) (\frac{dh}{dy})^T \tag{50}$$

$$Cov(\hat{\eta}, \hat{\zeta}) = (\frac{dg}{dy}) V(y) (\frac{dh}{dy})^T \tag{51}$$

$$V(\hat{\eta}) = V(y) = V(y)[I_N - (G - HUH^T)V(y)] \tag{81}$$

$$\begin{aligned} V(\hat{\zeta}) &= UH^TV(y)HU^T \\ &= (F_\zeta)^{-1}s(F_\zeta^T)^{-1}F_\zeta^T s^{-1}F_\eta V(y)F_\eta^T s^{-1}F_\zeta F_\zeta^{-1}s(F_\zeta^T)^{-1} \\ &= F_\zeta^{-1}s(F_\zeta^T)^{-1} = U \end{aligned} \tag{82}$$

$$Cov(\hat{\eta}, \hat{\zeta}) = -V(y)HU^T \tag{83}$$

$$= -V(y)HU \tag{84}$$

$$V(\hat{\eta}) = V(y) = V(y)[I_N - (G - HUH^T)V(y)] \tag{85}$$

$$V(\hat{\zeta}) = U \tag{86}$$

$$Cov(\hat{\eta}, \hat{\zeta}) = -V(y)HU \tag{87}$$

$$\frac{dg}{dy} = I_N \tag{76}$$

$$\frac{dh}{dy} = -UH^T \tag{77}$$

$$V(\hat{\eta}) = (\frac{dg}{dy})V(y)(\frac{dg}{dy})^T \tag{49}$$

$$V(\hat{\zeta}) = (\frac{dh}{dy})V(y)(\frac{dh}{dy})^T \tag{50}$$

$$Cov(\hat{\eta}, \hat{\zeta}) = (\frac{dg}{dy})V(y)(\frac{dh}{dy})^T \tag{51}$$

$$s = F_\eta V(y) F_\eta^T \tag{58}$$

$$G = F_\eta^T s^{-1} F_\eta,\ H = F_\eta^T s^{-1} F_\zeta,\ U^{-1} = F_\zeta^T s^{-1} F_\zeta \tag{57}$$

✓ Summary

- Expected value

$$\zeta^{v+1} = \zeta^{v} - [(F_\zeta^T)^v s^{-1} F_\zeta^v]^{-1} (F_\zeta^T)^v s^{-1} r \tag{27}$$

$$\lambda^{v+1} = s^{-1}[r + F_\zeta^v(\zeta^{v+1} - \zeta^v)] \tag{28}$$

$$\eta^{v+1} = y - V(y)(F_\eta^T)^v \lambda^{v+1} \tag{29}$$

- Statistical errors

$$V(\hat{\eta}) = (\frac{dg}{dy})V(y)(\frac{dg}{dy})^T \tag{49}$$

$$V(\hat{\zeta}) = (\frac{dh}{dy})V(y)(\frac{dh}{dy})^T \tag{50}$$

$$Cov(\hat{\eta}, \hat{\zeta}) = (\frac{dg}{dy})V(y)(\frac{dh}{dy})^T \tag{51}$$

$$V(\hat{\eta}) = V(y) = V(y)[I_N - (G - HUH^T)V(y)] \tag{85}$$

$$V(\hat{\zeta}) = U \tag{86}$$

$$Cov(\hat{\eta}, \hat{\zeta}) = -V(y)HU \tag{87}$$

$$s = F_\eta V(y) F_\eta^T \tag{58}$$

$$G = F_\eta^T s^{-1} F_\eta,\ H = F_\eta^T s^{-1} F_\zeta,\ U^{-1} = F_\zeta^T s^{-1} F_\zeta \tag{57}$$

在BESIII分析中的应用实例 (Xiu) Ying-Hui Guan *et al* 2013 *Chinese Phys. C* **37** 106201

介绍最小二乘法+外部约束，非线性（拉格朗日乘子法）

↓

得到χ^2的表示，求最小（偏微分方程，非线性：泰勒展开），得到对参数的估计

应用：

利用MC来测试这种方法，得到拟合结果 → 验证这种方法的可靠性

$$\chi^2 \equiv (\boldsymbol{y}-\boldsymbol{\eta})^{\mathrm{T}}\boldsymbol{V}_{\boldsymbol{y}}^{-1}(\boldsymbol{y}-\boldsymbol{\eta})+2\boldsymbol{\lambda}_{\alpha}{}^{\mathrm{T}}\boldsymbol{g}(\boldsymbol{\eta},\boldsymbol{m})$$

$$+2\boldsymbol{\lambda}_{\beta}{}^{\mathrm{T}}\boldsymbol{h}(\boldsymbol{\eta}),$$

- $\boldsymbol{y}$ ：$\boldsymbol{\eta}$的一组观测值　$\boldsymbol{\eta}$:观测值相应的真值（期望值）$\boldsymbol{m}$ ： 感兴趣的未知参数
- $\boldsymbol{g}(\boldsymbol{\eta},\boldsymbol{m})$ 描述了$\boldsymbol{\eta}$ 和 $\boldsymbol{m}$ 的关系
- $\boldsymbol{h}(\boldsymbol{\eta})$ ： $\boldsymbol{\eta}$的约束函数

- $\boldsymbol{V_y}$ ： 观测值y的协方差矩阵，由测量得到，在拟合中通常认为是常数（也有些与$\boldsymbol{m}$有关），在这个研究中认为$\boldsymbol{V_y}$ 与$\boldsymbol{m}$有关：$\boldsymbol{V_y}(\boldsymbol{m})$，每次迭代都会更新
- $\boldsymbol{\lambda_\alpha}$ $\boldsymbol{\lambda_\beta}$ ： 拉格朗日乘数

- 在高能物理实验中常用的直接观测量：信号事例数$\boldsymbol{n}$，通常$\boldsymbol{n}$会受到其它信号过程以及本底过程的影响，这里用$\boldsymbol{b}$来表示peaking本底，经过效率修正，事例数用$\boldsymbol{c}$来表示：$\boldsymbol{c}=\boldsymbol{E}^{-1}\boldsymbol{s}=\boldsymbol{E}^{-1}(\boldsymbol{n}-\boldsymbol{F}\boldsymbol{b})$，其中

  $\boldsymbol{E}$ 是信号效率矩阵，描述的是探测效率以及各个感兴趣的信号过程之间相互的对效率的影响

  $\boldsymbol{F}$ 是本底效率矩阵，描述的是本底对各个信号过程效率的污染

  效率矩阵不一定是对角的

- $\boldsymbol{t}$ 是外部测量量，约束$\boldsymbol{m}$，可以将$\boldsymbol{c}$ $\boldsymbol{t}$ 都写在$\boldsymbol{y}$中

- 如果$\boldsymbol{g}(\boldsymbol{\eta},\boldsymbol{m})$是非线性的，需要进行泰勒展开，$\boldsymbol{h}(\boldsymbol{\eta})$同理

$$\boldsymbol{g}(\boldsymbol{\eta},\boldsymbol{m})\approx\boldsymbol{g}(\boldsymbol{\eta}_0,\boldsymbol{m}_0)+\frac{\partial\boldsymbol{g}}{\partial\boldsymbol{m}}(\boldsymbol{m}-\boldsymbol{m}_0)+\frac{\partial\boldsymbol{g}}{\partial\boldsymbol{\eta}}(\boldsymbol{\eta}-\boldsymbol{\eta}_0)$$

$$\boldsymbol{h}(\boldsymbol{\eta})\approx\boldsymbol{h}(\boldsymbol{\eta}_0)+\frac{d\boldsymbol{h}}{d\boldsymbol{\eta}}(\boldsymbol{\eta}-\boldsymbol{\eta}_0)$$

误差

$$V_c = \left(\frac{\partial c}{\partial n}\right)^{\mathrm{T}} V_n \frac{\partial c}{\partial n} + \left(\frac{\partial c}{\partial b}\right)^{\mathrm{T}} V_b \frac{\partial c}{\partial b}$$
$$+ \left(\left(\frac{\partial c}{\partial E}\right)^{\mathrm{T}} \left(\frac{\partial c}{\partial F}\right)^{\mathrm{T}} \right) \begin{pmatrix} V_E & C_{EF} \\ C_{EF}^{\mathrm{T}} & V_F \end{pmatrix} \begin{pmatrix} \frac{\partial c}{\partial E} \\ \frac{\partial c}{\partial F} \end{pmatrix}$$

- 误差传递 $\boldsymbol{c} = \boldsymbol{E}^{-1}\boldsymbol{s} = \boldsymbol{E}^{-1}(\boldsymbol{n} - \boldsymbol{F}\boldsymbol{b})$
- 对于不同过程，观测量的误差是相关的，所以$\boldsymbol{V_E}$ $\boldsymbol{V_F}$有非零的对角元
- 通常外部测量量$\boldsymbol{t}$和内部测量量$\boldsymbol{c}$之间是不相关的，故 $\boldsymbol{V_y} = \begin{pmatrix} \boldsymbol{V_c} & 0 \\ 0 & \boldsymbol{V_t} \end{pmatrix}$，如果 $\boldsymbol{c}$ $\boldsymbol{t}$相关，对角元也会是非零的

求χ^2最小值

- 迭代：第k步估计值$\boldsymbol{m}^k$是用来计算第k+1步估计值$\boldsymbol{m}^{k+1}$的“seed”，每次迭代，受$\boldsymbol{m}$影响的$\boldsymbol{V_c}$ $\boldsymbol{c}$ 等变量都会更新
- 重复(7)式，直到找到最小值（χ^2稳定地收敛在某个值附近）

省略一些小项$\frac{\partial \boldsymbol{c}}{\partial \boldsymbol{m}}$ $\frac{\partial \boldsymbol{V_c}}{\partial \boldsymbol{m}}$来避免bias

$$\begin{aligned}\boldsymbol{m}^{k+1} &= \boldsymbol{m}^k - [\boldsymbol{G}_{\boldsymbol{m}}^k \boldsymbol{S}_4^{-1} (\boldsymbol{G}_{\boldsymbol{m}}^k)^{\mathrm{T}}]^{-1} \boldsymbol{G}_{\boldsymbol{m}}^k \boldsymbol{S}_4^{-1} \\ &\quad \times [\boldsymbol{z}_1 - (\boldsymbol{G}_{\boldsymbol{\eta}}^{\mathrm{T}})^k \boldsymbol{V}_{\boldsymbol{y}} (\boldsymbol{H}_{\boldsymbol{\eta}})^k \boldsymbol{S}_2^{-1} \boldsymbol{z}_2],\end{aligned} \tag{7}$$

where

$$\begin{aligned}(\boldsymbol{G_m})_{il} \equiv \frac{\partial g_l}{\partial m_i},\ (\boldsymbol{G_\eta})_{jl} \equiv \frac{\partial g_l}{\partial \eta_j}, \\ (\boldsymbol{H_m})_{il} \equiv \frac{\partial h_l}{\partial m_i},\ (\boldsymbol{H_\eta})_{jl} \equiv \frac{\partial h_l}{\partial \eta_j},\end{aligned} \tag{8}$$

$$\begin{aligned}\boldsymbol{S}_1 \equiv (\boldsymbol{G}_{\boldsymbol{\eta}}^{\mathrm{T}})^k \boldsymbol{V}_{\boldsymbol{y}} \boldsymbol{G}_{\boldsymbol{\eta}}^k, \\ \boldsymbol{S}_2 \equiv (\boldsymbol{H}_{\boldsymbol{\eta}}^{\mathrm{T}})^k \boldsymbol{V}_{\boldsymbol{y}} \boldsymbol{H}_{\boldsymbol{\eta}}^k\end{aligned} \tag{9}$$

$$\begin{aligned}&\boldsymbol{S}_3 \equiv (\boldsymbol{G}_{\boldsymbol{\eta}}^{\mathrm{T}})^k \boldsymbol{V}_{\boldsymbol{y}} \boldsymbol{H}_{\boldsymbol{\eta}}^k \boldsymbol{S}_2^{-1} (\boldsymbol{H}_{\boldsymbol{\eta}}^{\mathrm{T}})^k \boldsymbol{V}_{\boldsymbol{y}} (\boldsymbol{G}_{\boldsymbol{\eta}}^k), \\ &\boldsymbol{S}_4 \equiv \boldsymbol{S}_1 - \boldsymbol{S}_3,\end{aligned} \tag{10}$$

$$\begin{aligned}\boldsymbol{z}_1 \equiv \boldsymbol{g}^k + \boldsymbol{G}_{\boldsymbol{\eta}}^{\mathrm{T}} (\boldsymbol{y} - \boldsymbol{\eta}^k), \\ \boldsymbol{z}_2 \equiv \boldsymbol{h}^k + \boldsymbol{H}_{\boldsymbol{\eta}}^{\mathrm{T}} (\boldsymbol{y} - \boldsymbol{\eta}^k).\end{aligned} \tag{11}$$

利用$e^+e^- \to \psi(3770) \to D\bar{D}$ 过程的MC 来测试

约束函数

y

- 感兴趣的变量 $D^0-\bar{D}^0$ mixing parameters

D decay mode	f^{cor}
$K^-\pi^+$	$1+R_{WS}$
K^+K^-	2
$K_S\pi^0$	2
$K^-\pi^+$, $K^+\pi^-$	$(1+R_{WS})^2-4r\cos\delta_{K\pi}(r\cos\delta_{K\pi}+y_D)$
$K^-\pi^+$, K^+K^-	$1+R_{WS}+2r\cos\delta_{K\pi}+y_D$
$K^-\pi^+$, $K_S\pi^0$	$1+R_{WS}-2r\cos\delta_{K\pi}-y_D$
$K^-\pi^+$, $K^+e^-\bar{\nu}_e$	$1-ry_D\cos\delta_{K\pi}-rx_D\sin\delta_{K\pi}$
K^+K^-, $K_S\pi^0$	4
K^+K^-, $Ke\nu_e$	$2(1+y_D)$
$K_S\pi^0$, $Ke\nu_e$	$2(1-y_D)$

c	t	m	relationship
c_i	R_{WS} r^2 $\delta_{K\pi}$ x_D y_D x'^2 y'	N_{DD} $\mathcal{B}(KK)$ $\mathcal{B}(K_S\pi^0)$ $\mathcal{B}(K\pi)$ $\mathcal{B}(Ke\nu)$ r	$R_{WS}=r^2+ry_D\cos(\delta_{K\pi})-rx_D\sin(\delta_{K\pi})+\frac{x_D^2+y_D^2}{2}$, $x'=x_D\cos\delta_{K\pi}+y_D\sin\delta_{K\pi}$, $y'=y_D\cos\delta_{K\pi}-x_D\sin\delta_{K\pi}$.

R_{WS} : wrong-sign rate ratio,

$R_{WS} \equiv \Gamma(\bar{D}^0 \to K^-\pi^+)/(D^0 \to K^-\pi^+)$

x'^2 : $x'^2 \equiv 2R_M\, y'^2$

y'^2 : $y' \equiv y\cos\delta - x\sin\delta$

假设这些外部输入变量之间的误差是不相关的

- 拟合过程中，9 个参数会更新：

N_{DD} 产生的$D^0-\bar{D}^0$对的总数 外部输入的值（其它外部输入值：world-average values）

r 振幅比 $<K^+\pi^-|D^0>/<K^+\pi^-|\bar{D}^0> \equiv re^{i\delta_{K\pi}}$

$\delta_{K\pi}$ 双Cabibbo压低过程$D^0 \to K^+\pi^-$的振幅与Cabibbo-favored过程$\bar{D}^0 \to K^+\pi^-$的振幅的相对相位

x_D y_D 描述charm mixing的参数

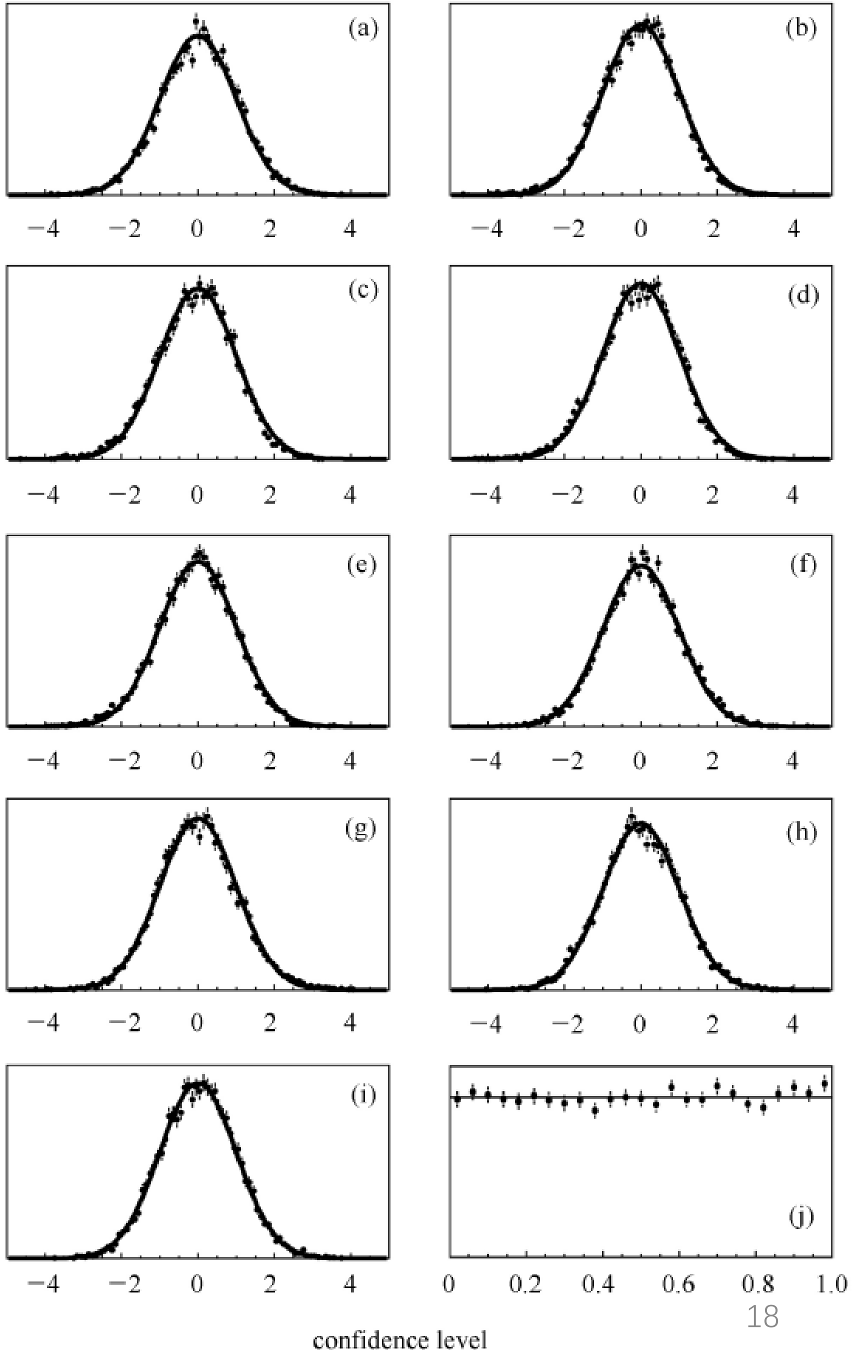
拟合结果

- 9个拟合参数的pull分布：可见都与正态分布符合得很好
- 结果说明这种方法给出了无偏估计以及正确的误差，说明了这种方法是可靠的，pull分布中轻微的不对称是由于非线性造成的

Table 3. Correlation coefficients, including systematic uncertainties, for the parameters determined by the fit with MC samples.

	N_{DD}	$\mathcal{B}(K\pi)$	$\mathcal{B}(KK)$	$\mathcal{B}(K_S\pi^0)$	$\mathcal{B}(Ke\nu)$	r	$\delta_{K\pi}$	x_D	y_D
N_{DD}	1	−0.63	−0.63	−0.24	−0.09	−0.02	0.03	−0.01	−0.01
$\mathcal{B}(K\pi)$		1	0.96	0.15	0.65	0.01	−0.02	0.00	0.01
$\mathcal{B}(KK)$			1	0.15	0.63	0.01	−0.02	0.00	0.01
$\mathcal{B}(K_S\pi^0)$				1	−0.03	0.01	−0.02	0.00	0.01
$\mathcal{B}(Ke\nu)$					1	−0.00	0.00	−0.00	0.01
r						1	0.06	0.11	−0.28
$\delta_{K\pi}$							1	−0.09	0.09
x_D								1	−0.09
y_D									1

- 相关系数，可见分支比彼此是正相关的，分支比与N_{DD}是负相关的



总结

- Global fit 可以考虑各个道之间的关联性，结合实验测量值和外部输入值，来得到预期的参数
- 效率矩阵不一定是对角的，所以一个道的MC估计情况会影响到整个效率矩阵
- 观测量和放入的误差矩阵认为是依赖于所要估计的参数，并且每次迭代都会被更新。当输入正确的观测量的误差矩阵时，拟合就能得到无偏估计，以及待估参数的正确的误差。

backup

Charm mixing in the Standard Model is conventionally described by two small dimensionless parameters:

$$x \equiv 2\frac{M_2 - M_1}{\Gamma_2 + \Gamma_1}, \tag{1}$$

$$y \equiv \frac{\Gamma_2 - \Gamma_1}{\Gamma_2 + \Gamma_1}, \tag{2}$$

where $M_{1,2}$ and $\Gamma_{1,2}$ are the masses and widths, respectively, of the neutral D meson CP eigenstates, D_1 (CP-odd) and D_2 (CP-even), defined by

$$|D_1\rangle \equiv \frac{|D^0\rangle + |\bar{D}^0\rangle}{\sqrt{2}}, \tag{3}$$

$$|D_2\rangle \equiv \frac{|D^0\rangle - |\bar{D}^0\rangle}{\sqrt{2}}, \tag{4}$$

assuming CP conservation. The mixing probability is then denoted by $R_{\rm M} \equiv (x^2 + y^2)/2$, and the width of the D^0 and $\bar{D}^0$ flavor eigenstates is $\Gamma \equiv (\Gamma_1 + \Gamma_2)/2$.