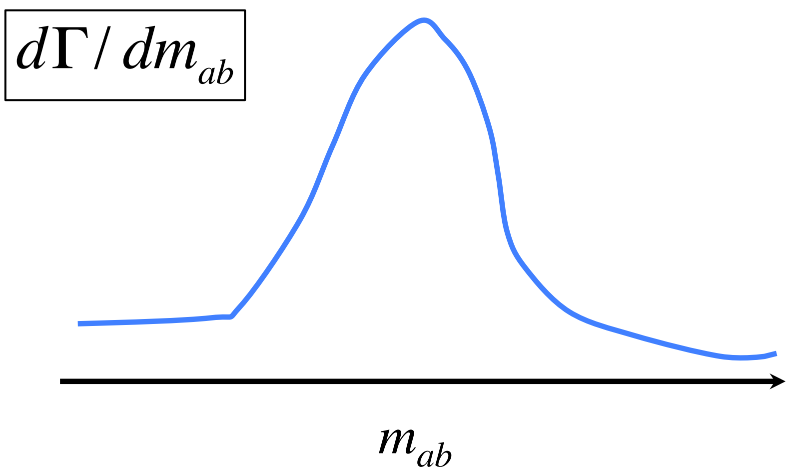
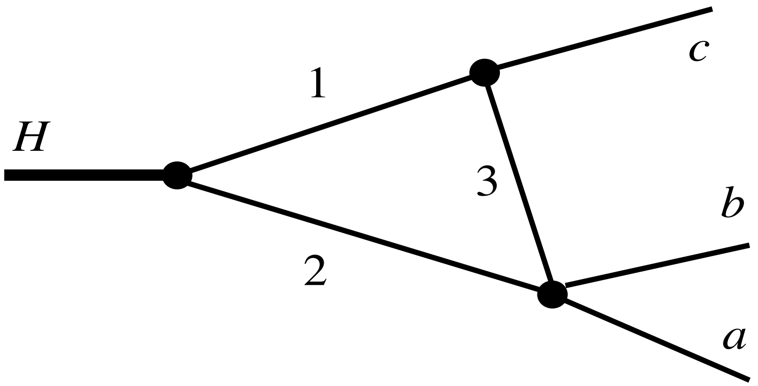
Basics of
Triangle Singularity

Triangle Singularity (TS)

My textbook:
Bayar, Aceti, Guo, Oset
PRD 94, 074039 (2016)

H → *abc* decay process

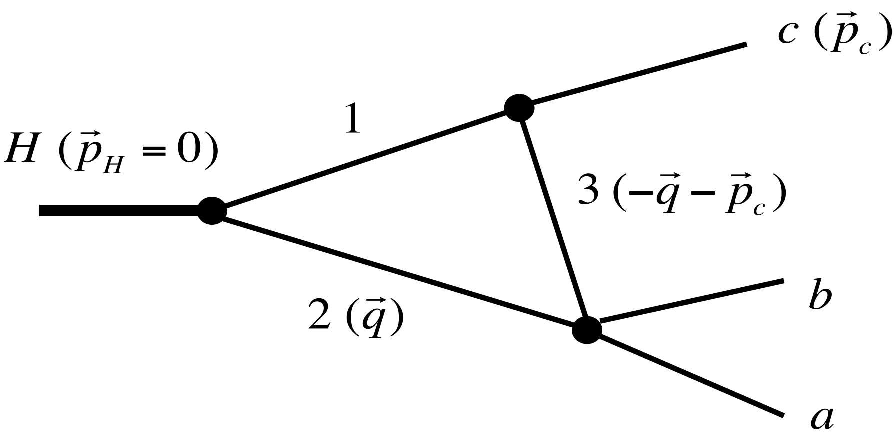




This triangle diagram can also create a resonance-like spectrum shape

if a special kinematical condition is satisfied

(The peak is not necessarily due to resonance)



All vertices are s-wave and coupling = 1

$$A(m_{ab}) \sim \int d^3q \, \frac{1}{E - E_2 - E_3 - E_c + i\varepsilon} \, \frac{1}{E - E_1 - E_2 + i\varepsilon} \qquad (E = m_H)$$

Non-relativistic energy (μ_{12} : reduced mass of 1 and 2; $p_c = p_c(m_{ab})$)

$$\sim \int d^3q \, \frac{1}{\underbrace{E - E_c - m_2 - m_3}_{\equiv E'} - \dfrac{q^2}{2m_2} - \dfrac{(-\vec{q} - \vec{p}_c)^2}{2m_3} + i\varepsilon} \, \frac{1}{\underbrace{E - m_1 - m_2}_{\equiv E''} - \dfrac{q^2}{2\mu_{12}} + i\varepsilon}$$

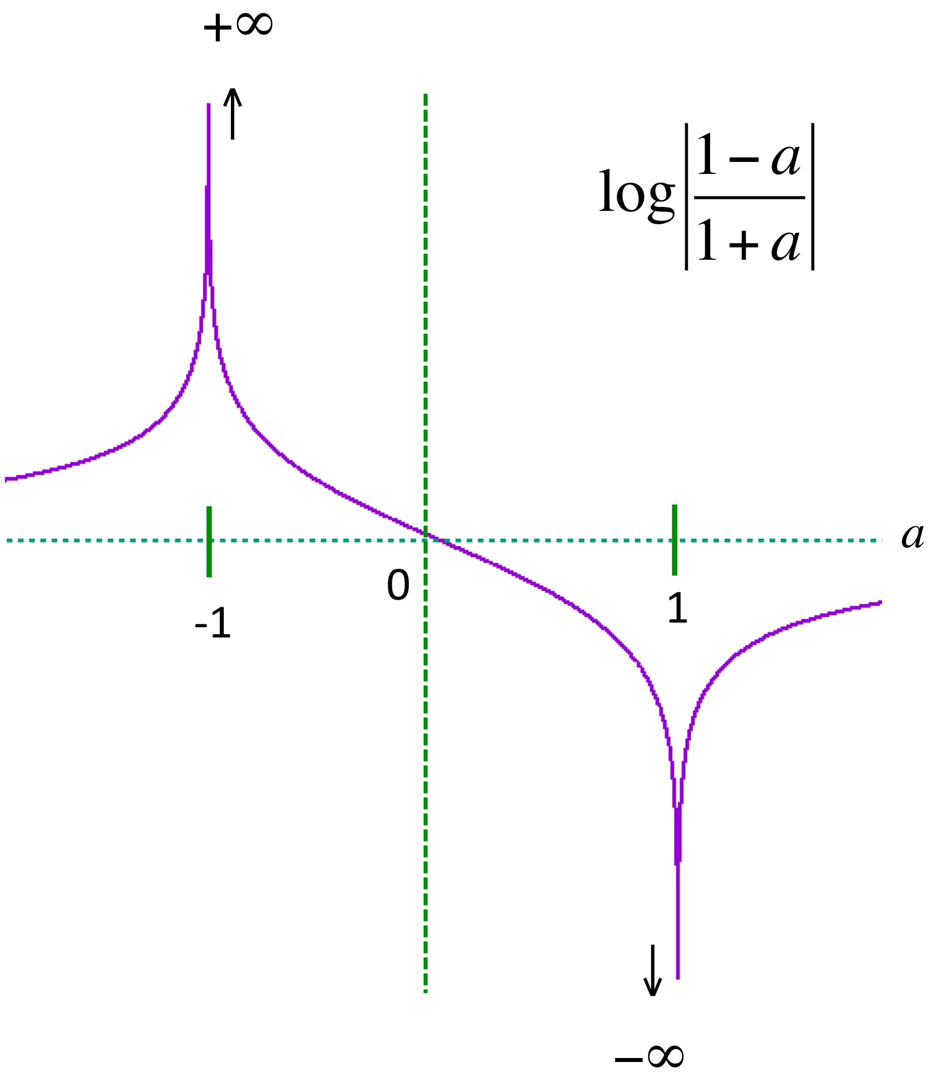
$$A = \int d^3q \, \frac{1}{\underbrace{E - E_c - m_2 - m_3}_{\equiv E'} - \frac{q^2}{2m_2} - \frac{(-\vec{q} - \vec{p}_c)^2}{2m_3} + i\varepsilon} \, \frac{1}{\underbrace{E - m_1 - m_2}_{\equiv E''} - \frac{q^2}{2\mu_{12}} + i\varepsilon}$$

$$= 2\pi \int_0^\infty q^2 \, dq \int_{-1}^{1} d\cos\overset{\hat{q}\cdot\hat{p}_c}{\theta} \, \frac{1}{E' - \frac{q^2}{2m_2} - \frac{q^2 + p_c^2}{2m_3} - \frac{qp_c}{m_3}\cos\theta + i\varepsilon} \, \frac{1}{E'' - \frac{q^2}{2\mu_{12}} + i\varepsilon}$$

$$= 2\pi \int_0^\infty q^2 \, dq \left(-\frac{m_3}{qp_c} \right) \left[\log\left| \frac{1-a}{1+a} \right| + i\pi\theta(1 - |a|) \right] \frac{1}{E'' - \frac{q^2}{2\mu_{12}} + i\varepsilon}$$

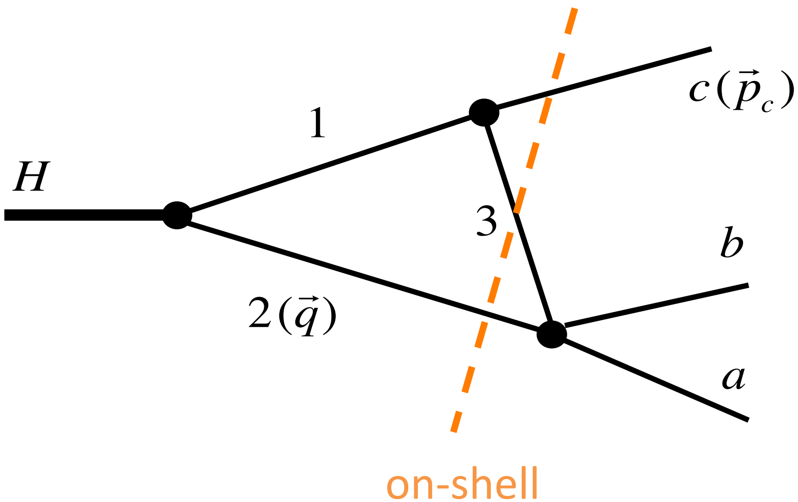
with $$a \equiv \frac{m_3}{qp_c} \left(E' - \frac{q^2}{2m_2} - \frac{q^2 + p_c^2}{2m_3} \right)$$

Logarithmic singularity



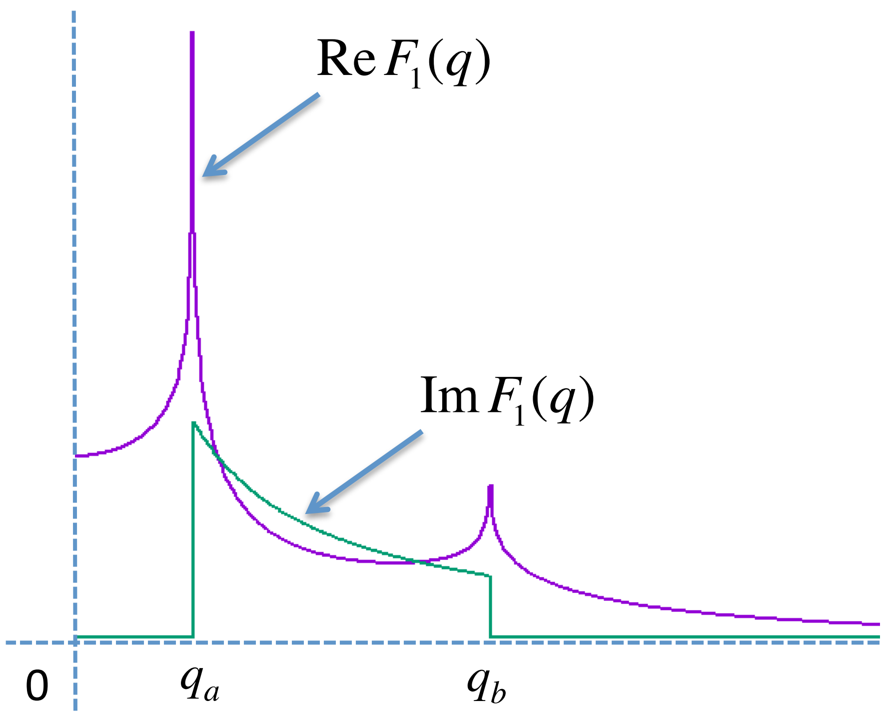
Logarithmic singularity occurs at $a = \pm 1$

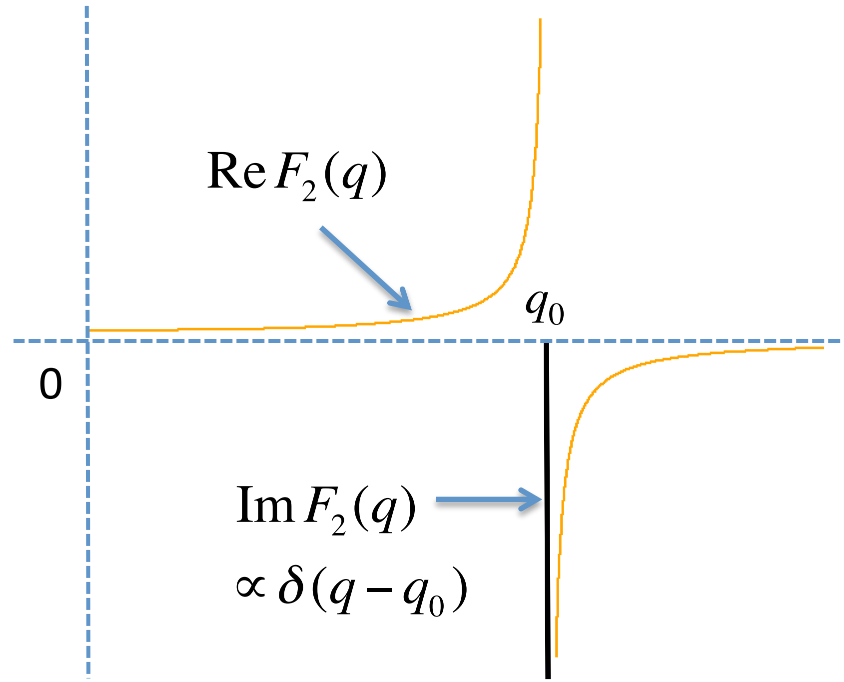
$$\Leftrightarrow \quad \hat{q}\cdot\hat{p}_c = \pm 1 \quad \text{and} \quad E = E_2 + E_3 + E_c$$

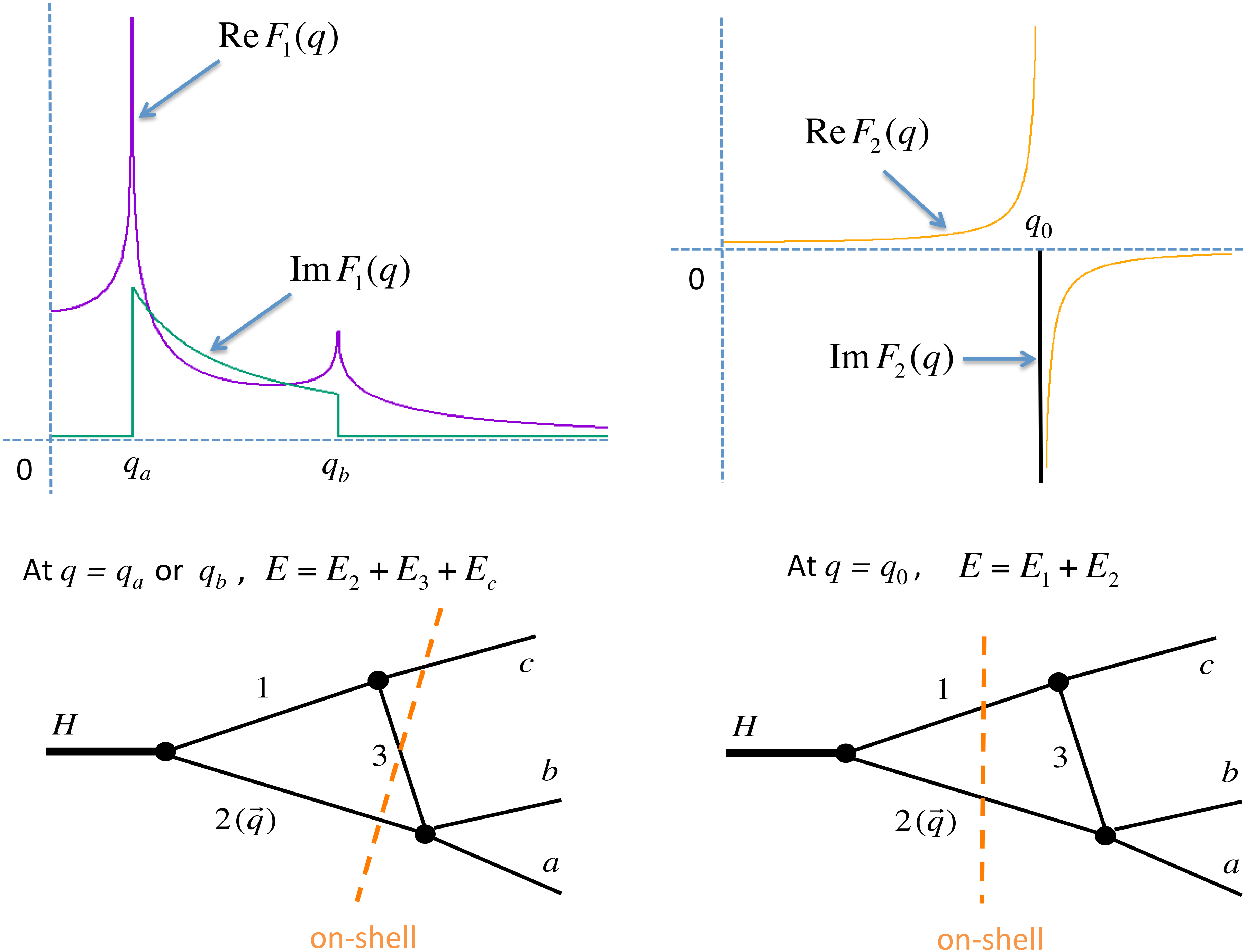


Momenta of particles 1, 2, 3, c are collinear

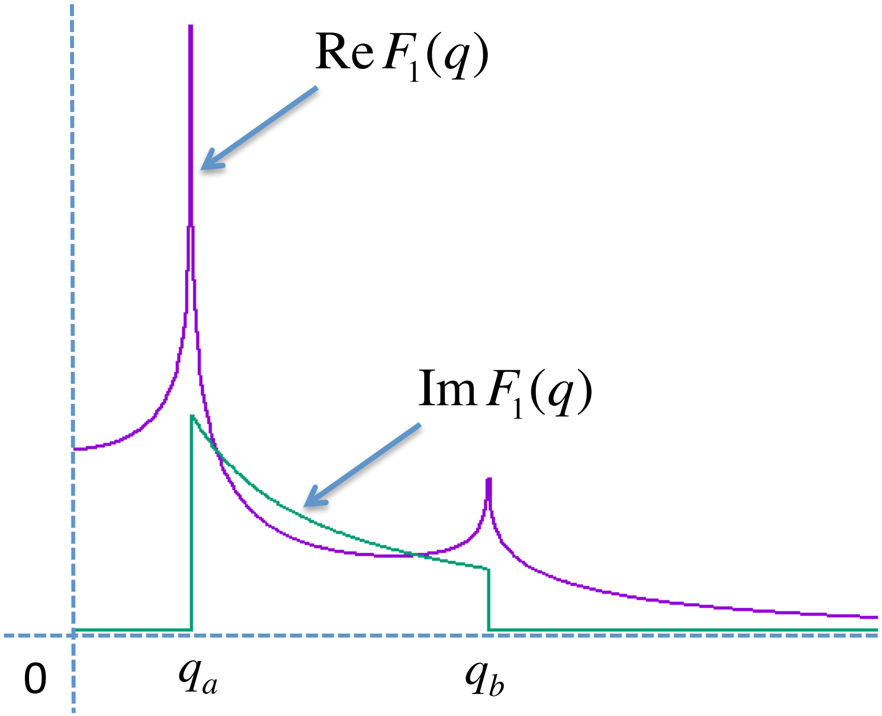
$$A(m_{ab}) = 2\pi \int_0^\infty q^2\, dq\, \underbrace{\left(-\frac{m_3}{q p_c}\right)\left[\log\left|\frac{1-a}{1+a}\right| + i\pi\theta(1-|a|)\right]}_{\equiv -F_1(q)} \underbrace{\frac{1}{E'' - \dfrac{q^2}{2\mu_{12}} + i\varepsilon}}_{\equiv F_2(q)}$$

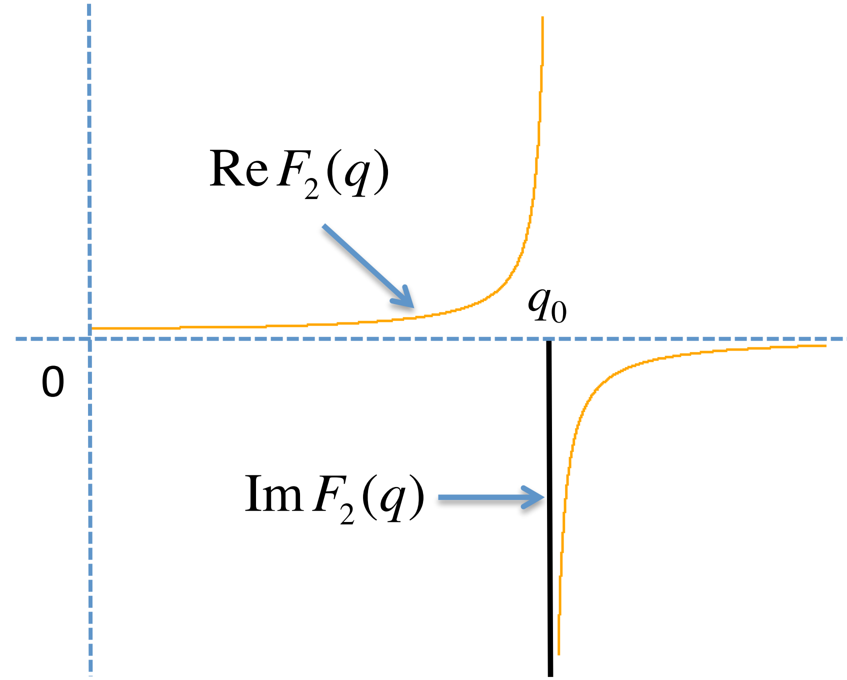






Re $F_1(q)$
Im $F_1(q)$
0
q_a
q_b
Re $F_2(q)$
Im $F_2(q)$
q_0
0
At $q = q_a$ or q_b , $E = E_2 + E_3 + E_c$
At $q = q_0$, $E = E_1 + E_2$
H
1
3
$2(\vec{q})$
c
b
a
on-shell
H
1
3
$2(\vec{q})$
c
b
a
on-shell

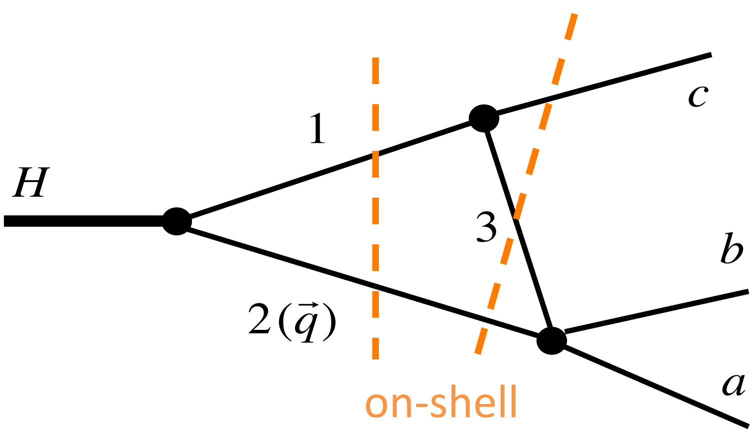




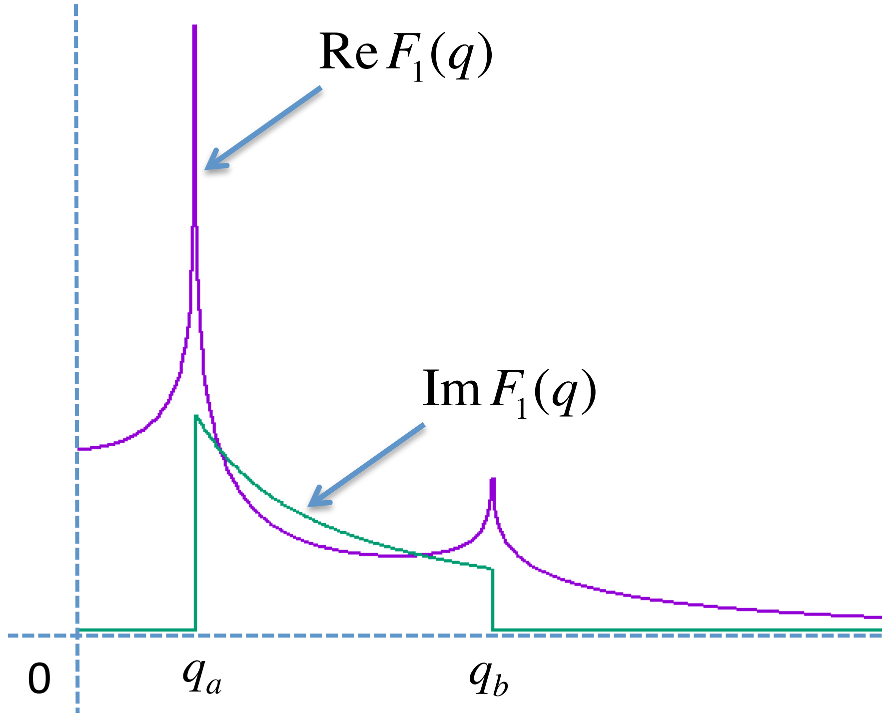
At $q = q_a$ or q_b , $E = E_2 + E_3 + E_c$

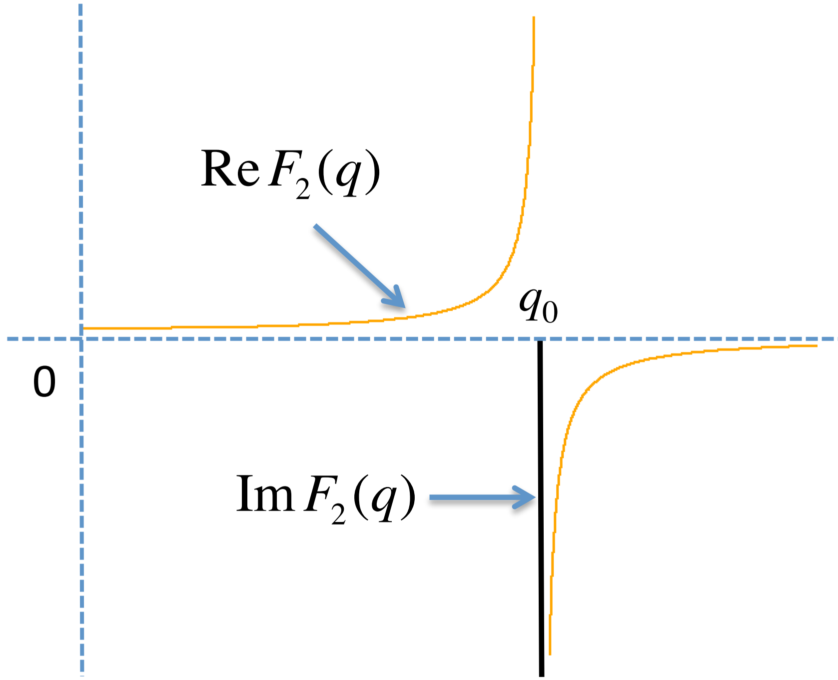
At $q = q_0$, $E = E_1 + E_2$

At special kinematical condition $q = q_\mathrm{a} = q_0$, all intermediate states are on-shell simultaneously



and $F_1(q)$ and $F_2(q)$ are singular simultaneously





At special kinematical condition $q = q_a = q_0$, all intermediate states are on-shell simultaneously

and $F_1(q)$ and $F_2(q)$ are singular simultaneously

Triangle singularity

$$A(m_{ab}) \propto \int_0^\infty q^2\,dq\;\; F_1(q)\,F_2(q) \;\longrightarrow \infty \qquad \text{at} \quad m_{ab} = m_{ab}^{\mathrm{TS}} \quad (\text{where } q_a = q_0)$$

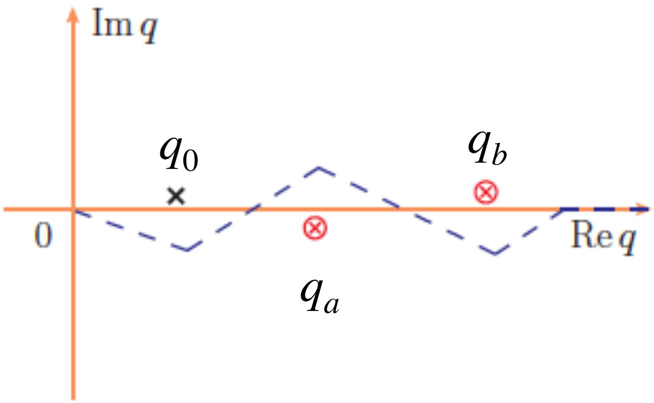
$q_b = q_0$ case: no TS occurs : classically forbidden kinematics (Coleman-Norton theorem)

all intermediate states are on-shell, but particle 3 cannot catch up with particle 2

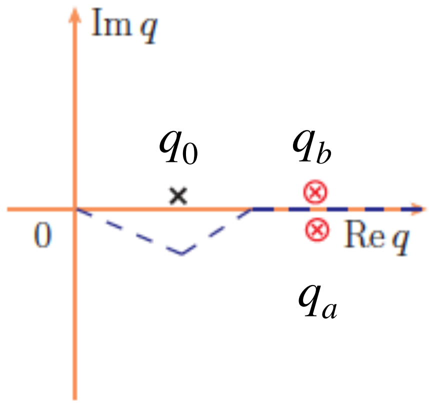
q-path (loop integral) in complex momentum plane and singularities from $F_1(q)$ $F_2(q)$

q_a, q_b : logarithmic singularity points of $F_1(q)$

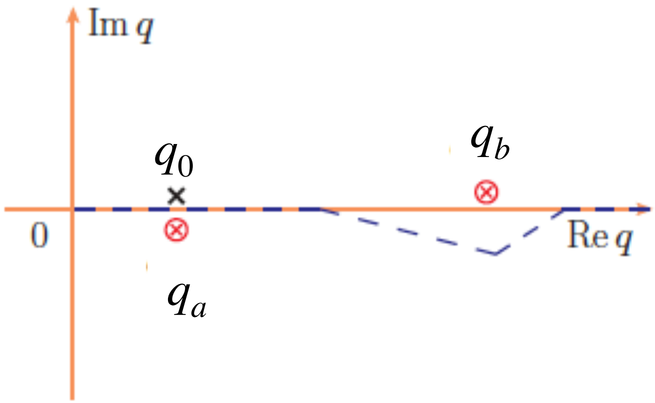
q_0 : pole of $F_2(q)$



(a) No singularity



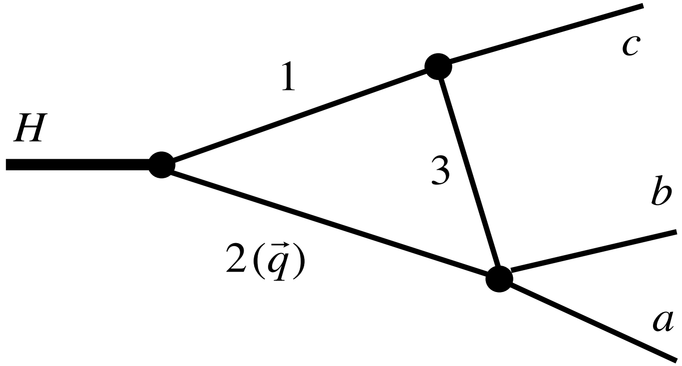
(b) Threshold cusp



(c) Triangle singularity

Triangle amplitude gets singular only when the q-path is pinched by two singularities

← no deformed path available to avoid singularities

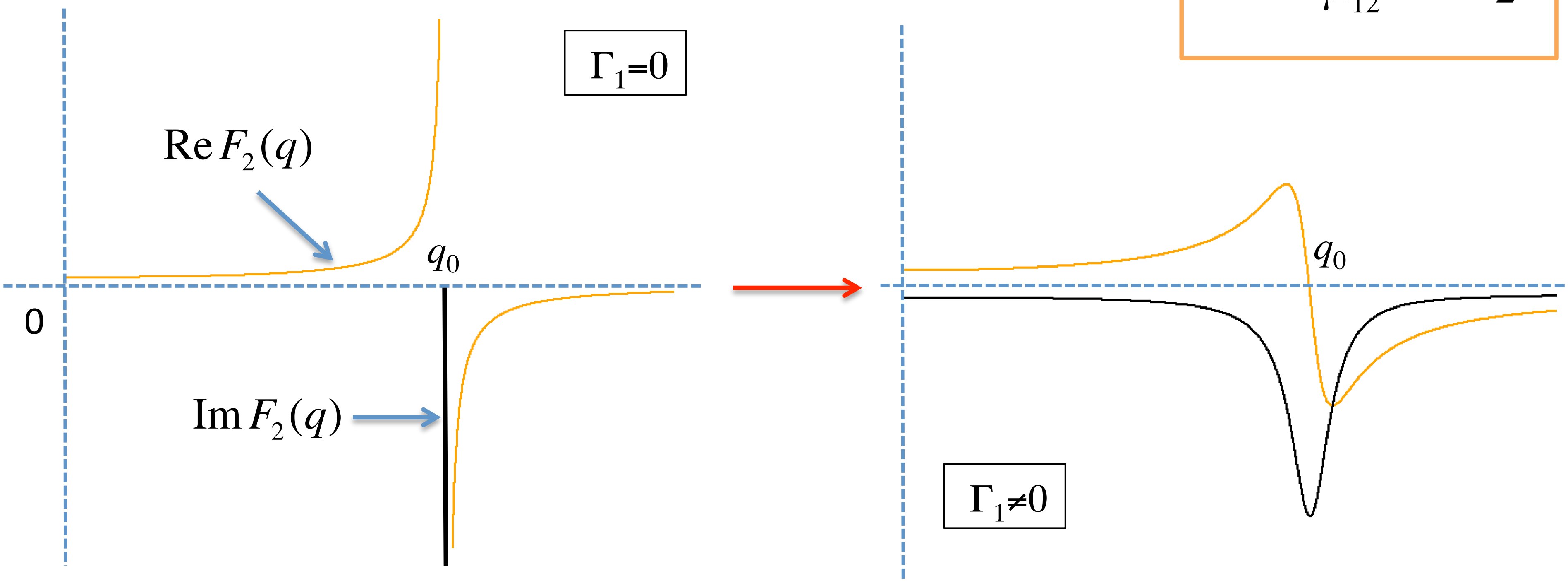


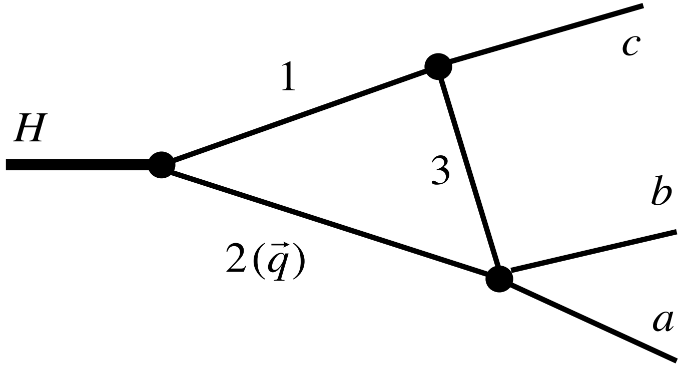
In reality,

particle 1 decays to particles 3 and c

→ particle 1 has a finite width

$$A(m_{ab}) = 2\pi \int_0^\infty q^2\, dq \left(-\frac{m_3}{q p_c}\right)\left[\log\left|\frac{1-a}{1+a}\right| + i\pi\theta(1-|a|)\right] \underbrace{\boxed{\frac{1}{E'' - \dfrac{q^2}{2\mu_{12}} + i\cancel{\epsilon}\dfrac{\Gamma_1}{2}}}}_{\equiv F_2(q)}$$





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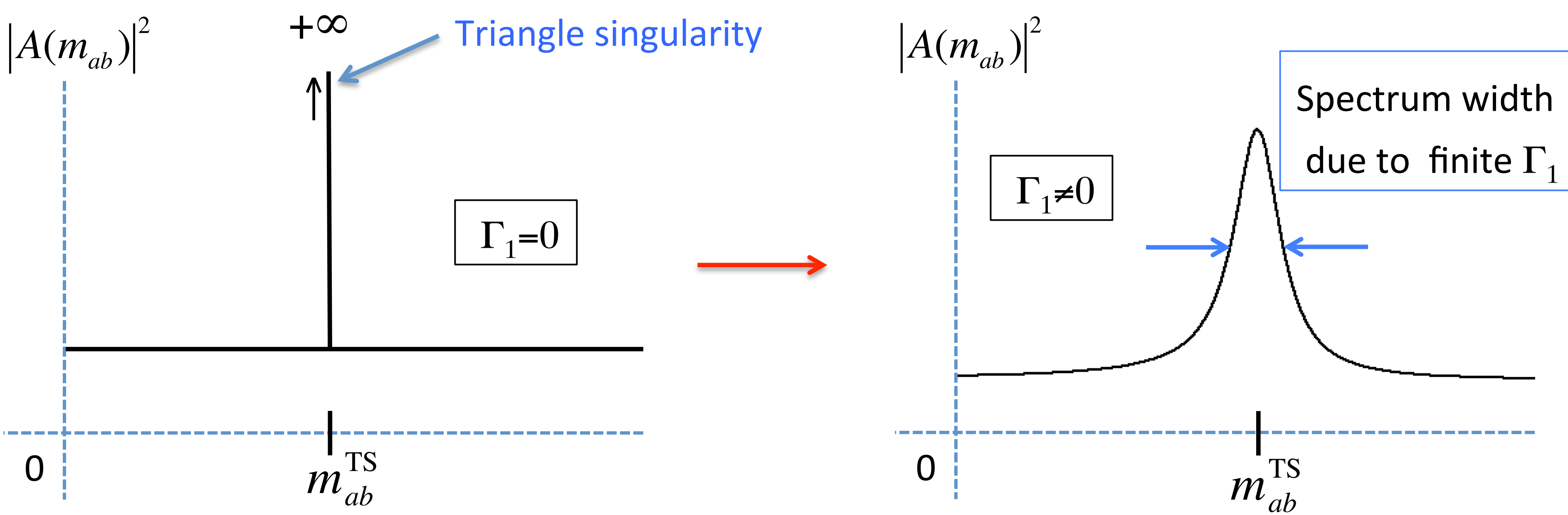
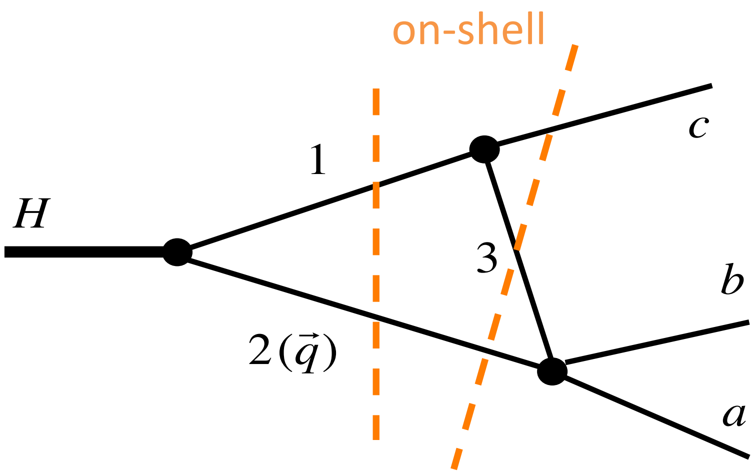


Short summary of TS



TS occurs in small kinematical window

where the process is kinematically allowed at classical level

- All intermediate states are on-shell
- All internal momenta are collinear (for $\vec{p}_H = 0$)
- Particle 3 catch up with particle 2

Triangle amplitude is significantly enhanced !